



Institut für Numerische Simulation

Rheinische Friedrich-Wilhelms-Universität Bonn

Wegelerstraße 6 • 53115 Bonn • Germany  
phone +49 228 73-3427 • fax +49 228 73-7527  
[www.ins.uni-bonn.de](http://www.ins.uni-bonn.de)

Dietmar Gallistl

**Stable splitting of polyharmonic operators by  
generalized Stokes systems**

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# STABLE SPLITTING OF POLYHARMONIC OPERATORS BY GENERALIZED STOKES SYSTEMS

DIETMAR GALLISTL

ABSTRACT. A stable splitting of  $2m$ -th order elliptic partial differential equations into  $2(m-1)$  problems of Poisson type and one generalized Stokes problem is established for any space dimension  $d \geq 2$  and any integer  $m \geq 1$ . This allows a numerical approximation with standard finite elements that are suited for the Poisson equation and the Stokes system, respectively. For some fourth- and sixth-order problems in two and three space dimensions, precise finite element formulations along with a priori error estimates and numerical experiments are presented.

## 1. INTRODUCTION

Let  $\Omega \subseteq \mathbb{R}^d$  be an open, simply-connected and bounded polytope with Lipschitz boundary. This article deals with the numerical approximation of the solutions to partial differential equations (PDEs) of polyharmonic type involving the  $m$ -th power  $\Delta^m$  of the Laplace operator  $\Delta$  as highest derivative for an integer  $m \geq 1$ . The exact solution is customarily sought in an  $m$ -th order Sobolev space, for instance  $H_0^m(\Omega)$ . A model problem is to seek, for a given  $f \in L^2(\Omega)$ , a function  $u \in H_0^m(\Omega)$  such that

$$(1) \quad (-1)^m \Delta^m u = f \quad \text{in } H^{-m}(\Omega).$$

The numerical approximation of (1) with conforming finite elements, requires the trial functions to belong to the Sobolev space  $H_0^m(\Omega)$ . This means that the finite element functions must satisfy  $C^{m-1}$  continuity across the inter-element boundaries. One prominent instance of (1) is the biharmonic equation  $\Delta^2 u = f$  for  $m = 2$ . Even in this case and for two space-dimensions, the design and implementation of conforming finite elements [13] is a difficult task and there have been many different attempts to circumvent the use of conforming methods, such as nonconforming finite elements [13, 21] or discontinuous Galerkin methods [3, 15, 9]. For the case  $m \geq 3$ , besides the conforming finite elements of [29], the following approaches have been suggested: the  $C^0$  interior penalty method of [19] for  $m = 3$ , the nonconforming finite element method of [28] for arbitrary  $m$  with the restriction that  $m \geq d$ , and the mixed formulation based on Helmholtz-type decompositions [24, 25, 26] for the  $m$ -th Laplace operator with arbitrary  $m$ .

The method suggested in this work is based on a mixed system. The idea of mixed methods is to reduce a higher-order differential equation to a system of lower-order equations. It is, however, difficult to find stable mixed methods for  $m \geq 2$ : care has to be taken that the stability of the mixed formulation is independent of the regularity of the solution, see also the discussion in [7, §10.1.1]. In particular, some mixed methods for the biharmonic equation may be divergent in the presence of singularities, e.g., for simply-supported plates. This is related to the fact that

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the stability of those splittings requires additional regularity on the solution of the PDE. It is therefore desirable that the mixed formulation is stable, independent of regularity assumptions on the solution. This is the case for the splitting proposed in this paper. For the simplest case, the biharmonic equation, this formulation resembles the splitting of [11] for the Reissner-Mindlin model of moderately thick elastic plates, where a singularly perturbed version of a fourth-order problem is reduced to two Poisson equations and a Stokes system. The formulation shows in particular that the biharmonic equation in planar domains can be approximated with any stable finite element for the Stokes equations. It turns out that such an idea can be generalized to any order of differentiability  $m \geq 1$  and any space dimension  $d \geq 2$  leading to a generalized polyharmonic Stokes equation. The resulting splitting is natural in that it is derived from the principles of exterior calculus and symmetric tensors. In particular, high-order partial differential equations like (1) can be numerically approximated with standard, even low-order (independent of  $m$ ), finite elements for the Poisson and the classical Stokes problem. The method is presented for data  $f \in L^2(\Omega)$ , but the theory is also valid for the weaker assumption  $f \in H^{-1}(\Omega)$ . In order to keep the formalism to a minimum, the analysis is carried out for the splitting of the model problem (1). The application to more general elliptic systems is, however, possible; an outline is given in Subsection 4.5.

The remaining parts of this article are organized as follows. Section 2 introduces the notation on tensors and their differential operators and proves a result on the existence of generalized scalar potentials. In Section 3, the new split of high-order PDEs is presented. It relies on the study of the polyharmonic Stokes problem with symmetry constraint. Sections 4–5 present applications for the case for  $m = 2$  and  $m = 3$  in two and three space dimensions along with finite element formulations and a priori error estimates. Numerical computations are presented in Section 6. The remarks from Section 7 conclude the paper.

Standard notation on Lebesgue and Sobolev spaces applies throughout this article. A detailed description of differential operators in tensor-valued Sobolev spaces is given in Section 2. An inequality  $A \leq CB$  for some generic mesh-size independent constant  $C$  is denoted by  $A \lesssim B$  while  $A \approx B$  abbreviates  $A \lesssim B \lesssim A$ . Throughout this paper,  $m$  refers to the order of differentiability,  $n$  denotes the degree of a tensor, and  $d$  is the space dimension. The integration with respect to the  $d$ -dimensional Lebesgue measure is indicated by the symbol  $dx$  whereas  $ds$  denotes integration with respect to the  $(d-1)$ -dimensional Hausdorff measure; the integral mean is denoted by  $\bar{f}$ . If not specified otherwise,  $\Omega \subseteq \mathbb{R}^d$  is a bounded, open, simply-connected Lipschitz polytope with outer unit normal  $\nu$ . For a matrix  $A$  with transpose  $A^*$ , the symmetric and the skew-symmetric part are denoted by

$$\text{sym } A := \frac{1}{2}(A + A^*) \quad \text{and} \quad \text{skw } A := \frac{1}{2}(A - A^*).$$

The trace of  $A$  is denoted by  $\text{tr } A$ .

## 2. DIFFERENTIAL OPERATORS FOR TENSOR FIELDS

This section introduces the notation for tensor fields and states a result on generalized scalar potentials.

**2.1. Notation.** Let  $d \geq 2$ . For any nonnegative integer  $n \in \mathbb{N}_0$ , the set of  $n$ -tensors over  $\mathbb{R}^d$  is defined as

$$\mathbb{M}_d(n) := (\mathbb{R}^d)^n = \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{n \text{ times}} \quad \text{with the convention } (\mathbb{R}^d)^0 := \mathbb{R}.$$

In particular,  $\mathbb{M}_d(0) = \mathbb{R}$ ,  $\mathbb{M}_d(1) = \mathbb{R}^d$ ,  $\mathbb{M}_d(2) = \mathbb{R}^{d \times d}$ . Let  $\mathfrak{S}(n)$  denote the symmetric group of degree  $n$ . An element  $\tau \in \mathbb{M}_d(n)$  is referred to as symmetric, if any  $(j_1, \dots, j_n) \in \{1, \dots, d\}^n$  and any  $\sigma \in \mathfrak{S}(n)$  satisfy

$$\tau_{j_1, \dots, j_n} = \tau_{\sigma(j_1, \dots, j_n)}.$$

This defines the space  $\mathbb{S}_d(n)$  of symmetric  $n$ -tensors over  $\mathbb{R}^d$ , namely

$$\mathbb{S}_d(n) := \{\tau \in \mathbb{M}_d(n) : \tau \text{ is symmetric}\}.$$

In particular,  $\mathbb{S}_d(0) = \mathbb{R}$ ,  $\mathbb{S}_d(1) = \mathbb{R}^d$ , while  $\mathbb{S}_d(2)$  is the space of symmetric  $d \times d$  matrices.

The inner product of  $\sigma, \tau \in \mathbb{M}_d(n)$  is defined as

$$\sigma : \tau = \sum_{(j_1, \dots, j_n) \in \{1, \dots, d\}^n} \sigma_{j_1, \dots, j_n} \tau_{j_1, \dots, j_n}.$$

The space of  $\mathbb{M}_d(n)$ -valued functions over some domain  $\omega$  whose components are square integrable is denoted by  $L^2(\omega; \mathbb{M}_d(n))$ . The subspace of symmetric  $L^2$  tensor fields reads  $L^2(\omega; \mathbb{S}_d(n))$ . In the scalar case, abbreviate  $L^2(\omega) := L^2(\omega; \mathbb{R})$ . The  $L^2$  inner product and the  $L^2$  norm for fields  $\sigma, \tau \in L^2(\omega; \mathbb{M}_d(n))$  read

$$(\sigma, \tau)_{L^2(\omega)} := \int_{\omega} \sigma : \tau \, dx \quad \text{and} \quad \|\sigma\|_{L^2(\omega)} := \sqrt{(\sigma, \sigma)_{L^2(\omega)}}.$$

Let  $\Omega \subseteq \mathbb{R}^d$  be a simply-connected and bounded Lipschitz domain. For  $m \in \mathbb{N}_0$ , the  $m$ -th order Sobolev space of  $L^2(\Omega)$  functions whose generalized derivatives up to order  $m$  belong to  $L^2$  and whose boundary traces up to order  $(m-1)$  vanish, is denoted by  $H_0^m(\Omega)$ . Tensor fields with values in  $\mathbb{M}_d(n)$  (resp.  $\mathbb{S}_d(n)$ ) whose components belong to  $H_0^m(\Omega)$  are denoted by  $H_0^m(\Omega; \mathbb{M}_d(n))$  (resp.  $H_0^m(\Omega; \mathbb{S}_d(n))$ ). The following abbreviations will be used

$$V_{m,n}^{\mathbb{M}} := H_0^m(\Omega; \mathbb{M}_d(n)) \quad \text{and} \quad V_{m,n}^{\mathbb{S}} := H_0^m(\Omega; \mathbb{S}_d(n)).$$

The (total) derivative  $Dv \in V_{m-1, n+1}^{\mathbb{M}}$  of an element  $v \in V_{m,n}^{\mathbb{M}}$  has the representation

$$(Dv)_{j_1, \dots, j_n, j_{n+1}} = \partial_{j_{n+1}} v_{j_1, \dots, j_n}$$

for any  $(j_1, \dots, j_{n+1}) \in \{1, \dots, d\}^{n+1}$ . In other words,  $(Dv)_{j_1, \dots, j_n, \bullet}$  corresponds to the gradient of the function  $v_{j_1, \dots, j_n}$  and  $(Dv)_{j_1, \dots, j_{n-1}, \bullet, \bullet}$  corresponds to the Jacobian of the vector field  $v_{j_1, \dots, j_{n-1}, \bullet}$ .

Let  $n \geq 1$ . For any  $v \in V_{m,n}^{\mathbb{M}}$ , its component-wise exterior derivative  $\mathbf{rot} v \in V_{m-1, n+1}^{\mathbb{M}}$  is defined by

$$(\mathbf{rot} v)_{j_1, \dots, j_{n-1}, j_n, j_{n+1}} = \partial_{j_{n+1}} v_{j_1, \dots, j_{n-1}, j_n} - \partial_{j_n} v_{j_1, \dots, j_{n-1}, j_{n+1}}.$$

In other words, the  $d \times d$  matrix field (or 2-form)  $(\mathbf{rot} v)_{j_1, \dots, j_{n-1}, \bullet, \bullet}$  is the exterior derivative of the vector field (or 1-form)  $v_{j_1, \dots, j_{n-1}, \bullet}$ . If  $n = 0$ , that is if  $v$  is scalar-valued, then, by definition,  $\mathbf{rot} v = 0$ . This is consistent with the fact that the space of alternating 2-forms over  $\mathbb{R}$  is the trivial vector space  $\{0\}$ .

In particular, if  $v$  is a vector field ( $n = 1$ ) for  $d = 2$  or  $d = 3$ , the upper triangular part of the skew-symmetric matrix  $\mathbf{rot} v$  reads as

$$(2) \quad \mathbf{rot} v = \begin{pmatrix} 0 & \partial_2 v_1 - \partial_1 v_2 \\ & 0 \end{pmatrix} \quad \text{or} \quad \mathbf{rot} v = \begin{pmatrix} 0 & \partial_2 v_1 - \partial_1 v_2 & \partial_3 v_1 - \partial_1 v_3 \\ & 0 & \partial_3 v_2 - \partial_2 v_3 \\ & & 0 \end{pmatrix}$$

and can be identified with the usual ‘‘vector proxies’’

$$(3) \quad \mathbf{rot} v = \partial_2 v_1 - \partial_1 v_2 \quad \text{for } d = 2 \quad \text{or} \quad \mathbf{rot} v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} \quad \text{for } d = 3.$$

In order to distinguish between alternating quadratic forms and the vector representations, the vector representation of  $\mathbf{rot}$  is denoted by  $\mathit{rot}$  without boldface letters.

The divergence of a sufficiently smooth  $\mathbb{M}_d(n)$ -valued tensor field  $\tau$  has values in  $\mathbb{M}_d(n-1)$  and is defined via

$$(\operatorname{div} \tau)_{j_1, \dots, j_{n-1}} = \operatorname{tr}((D\tau)_{j_1, \dots, j_{n-1}, \bullet, \bullet}) = \sum_{k=1}^d \partial_k \tau_{j_1, \dots, j_{n-1}, k}.$$

The space of  $L^2(\Omega)$  vector fields whose divergence belongs to  $L^2(\Omega)$  is denoted by  $H(\operatorname{div}, \Omega)$ .

**2.2. Exterior calculus of symmetric tensor fields.** The following result serves as the main tool for the stable splitting of higher-order differential operators.

**Lemma 1.** *Let  $\Omega$  be simply-connected and let  $m \geq 1$  and  $n \geq 1$  be integers.*

- (a) *Let  $v \in V_{m,n}^{\mathbb{M}}$ . If  $\mathbf{rot} v = 0$  then there exists  $\eta \in V_{m+1, n-1}^{\mathbb{M}}$  such that  $D\eta = v$ .*
- (b) *Let  $v \in V_{m,n}^{\mathbb{M}}$ . The matrix  $(Dv)_{j_1, \dots, j_{n-1}, \bullet, \bullet}$  is symmetric for any index  $(j_1, \dots, j_{n-1}) \in \{1, \dots, d\}^{n-1}$  if and only if  $\mathbf{rot} v = 0$ .*
- (c) *Let  $v \in V_{m,n}^{\mathbb{S}}$ . If  $\mathbf{rot} v = 0$  then there exists  $\eta \in V_{m+1, n-1}^{\mathbb{S}}$  such that  $D\eta = v$ .*
- (d) *Let  $v \in V_{m,n}^{\mathbb{S}}$ . If  $\mathbf{rot} v = 0$ , then for any  $k \in \{1, \dots, n\}$  there exists  $\eta \in V_{m+k, n-k}^{\mathbb{S}}$  such that  $D^k \eta = v$ .*

*Proof.* Assertion (a) is a well-known fact from the exterior calculus on simply-connected domains. Assertion (b) is the observation that  $\mathbf{rot} v$  contains exactly the skew-symmetric parts of the matrices  $(Dv)_{j_1, \dots, j_{n-1}, \bullet, \bullet}$ . For the proof of (c), let  $v \in V_{m,n}^{\mathbb{S}}$ . From (a) it is known that there exists  $\eta \in V_{m+1, n-1}^{\mathbb{M}}$  such that  $D\eta = v$ . Let now  $\sigma \in \mathfrak{S}(n-1)$  be a permutation. The relation  $D\eta = v$  and the symmetry of  $v$  imply that for any  $(j_1, \dots, j_n) \in \{1, \dots, d\}^n$  the following identity holds

$$\partial_{j_n} \eta_{j_1, \dots, j_{n-1}} = v_{j_1, \dots, j_n} = v_{\sigma(j_1, \dots, j_{n-1}), j_n} = \partial_{j_n} \eta_{\sigma(j_1, \dots, j_{n-1})}.$$

This means that the difference  $\eta_{j_1, \dots, j_{n-1}} - \eta_{\sigma(j_1, \dots, j_{n-1})}$  is constant with respect to the variable  $x_{j_n}$  for any  $j_n \in \{1, \dots, d\}$ . From the boundary condition on  $\eta$  it turns out that  $\eta_{j_1, \dots, j_{n-1}} = \eta_{\sigma(j_1, \dots, j_{n-1})}$ . Hence,  $\eta$  is symmetric. Assertion (d) follows from the recursive application of (b) and (c).  $\square$

*Remark 2.* Symmetry principles like in Lemma 1 were also studied in [24, 25, 26] for the proof of generalized Helmholtz decompositions.

### 3. STABLE SPLITTING OF HIGHER-ORDER DIFFERENTIAL OPERATORS

In this section, the splitting of the  $2m$ -th order PDE in a system of second-order problems is presented. To keep the presentation simple, the polyharmonic operator  $(-1)^m \Delta^m$  without lower-order terms is analyzed. Subsection 4.5 will give an outline of the case of more general differential operators. The core part of the split consists of a polyharmonic generalized Stokes equation.

**3.1. The polyharmonic generalized Stokes equation.** Define the space of symmetric and rotation-free  $H_0^m$  tensor fields as

$$Z_{m,n} := \{v \in V_{m,n}^{\mathbb{S}} : \mathbf{rot} v = 0\}.$$

Let  $f \in L^2(\Omega; \mathbb{M}_d(n))$ . The polyharmonic generalized Stokes problem seeks  $u \in Z_{m,n}$  such that

$$(4) \quad (D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in Z_{m,n}.$$

This problem is referred to as generalized Stokes equation because it corresponds to the minimization of a quadratic energy functional subject to a linear constraint on the partial derivatives. Section 4–5 will show how the problem (4) can be rewritten as a saddle-point problem with Lagrange multipliers. Lemma 1(b) shows that  $Du$  is symmetric. With the substitution  $w := Du \in Z_{m-1,n+1}$  one arrives at the following alternative formulation: Seek  $u \in V_{1,n}^{\mathbb{S}}$ ,  $w \in Z_{m-1,n+1}$ ,  $r \in V_{1,n}^{\mathbb{S}}$  such that

$$(5a) \quad (Du, Dv)_{L^2(\Omega)} = (w, Dv)_{L^2(\Omega)} \quad \text{for all } v \in V_{1,n}^{\mathbb{S}},$$

$$(5b) \quad (D^{(m-1)}w, D^{(m-1)}z)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{m-1,n+1},$$

$$(5c) \quad (Dr, Ds)_{L^2(\Omega)} = (f, s)_{L^2(\Omega)} \quad \text{for all } s \in V_{1,n}^{\mathbb{S}}.$$

The unique solvability of (4) and (5) is a direct consequence of the obvious coercivity of the involved operators. The next result states the equivalence of (4) and (5).

**Proposition 3.** *If  $u \in Z_{m,n}$  solves (4), then there exists  $r \in V_{1,n}^{\mathbb{S}}$  such that  $(u, Du, r) \in V_{1,n}^{\mathbb{S}} \times Z_{m-1,n+1} \times V_{1,n}^{\mathbb{S}}$  solves (5). If, conversely,  $(u, w, r) \in V_{1,n}^{\mathbb{S}} \times Z_{m-1,n+1} \times V_{1,n}^{\mathbb{S}}$  solves (5), then  $u$  belongs to  $Z_{m,n}$  and solves (4) and  $w = Du$ .*

*Proof.* The sufficient condition is verified with the solution  $r$  of the Poisson-type problem (5c). Indeed,  $w := Du$  belongs to  $Z_{m-1,n+1}$  and, by Lemma 1(c), any test function  $z \in Z_{m-1,n+1}$  can be represented as  $z = Dv$  for some  $v \in V_{m,n}^{\mathbb{S}}$ . Hence,

$$(D^{(m-1)}w, D^{(m-1)}z)_{L^2(\Omega)} = (D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)}.$$

For the verification of the necessary condition, let  $(u, w, r)$  solve (5). By Lemma 1(c) there exists  $\eta \in V_{m,n}^{\mathbb{S}}$  such that  $D\eta = w$ . Since  $w$  is symmetric,  $\eta$  is rotation-free **rot**  $\eta = 0$  (cf. Lemma 1(b)). Hence,  $\eta \in Z_{m,n}$ . By testing equation (5a) with the test function  $v := u - \eta$ , one obtains

$$\|D(u - \eta)\|_{L^2(\Omega)}^2 = (D(u - \eta), Dv)_{L^2(\Omega)} = (Du - w, Dv)_{L^2(\Omega)} = 0.$$

This implies  $\eta = u$  and, thus,  $Du = w$ . According to Lemma 1(b), for any  $v \in Z_{m,n}$  the derivative  $Dv$  is symmetric. Hence, (5b) and (5c) lead to

$$(D^m u, D^m v)_{L^2(\Omega)} = (D^{(m-1)}w, D^{(m-1)}Dv)_{L^2(\Omega)} = (Dr, Dv)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

Therefore,  $u$  solves (4).  $\square$

System (5) consists of two Poisson equations and one generalized symmetric Stokes system. In the present case where no lower-order terms appear in the differential operator, the problem (5b) decouples from the equations (5a) and (5c). However, this decoupling does not hold in the presence of lower-order terms. Examples will be given below.

**3.2. Splitting of polyharmonic equations.** Let  $f \in L^2(\Omega)$  and consider the  $2m$ -th order partial differential equation that seeks  $u \in H_0^m(\Omega)$  with

$$(-1)^m \Delta^m u = f.$$

Its weak form is given by

$$(6) \quad (D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^m(\Omega).$$

The observation that  $H_0^m(\Omega) = V_{m,0}^{\mathbb{S}} = Z_{m,0}$  reveals that (6) is identical to problem (4) for  $n = 0$ . Hence, by applying recursively the split (5), one obtains a splitting of (6) into  $(2m-1)$  second-order equations, namely  $(2m-2)$  Poisson-like problems and one generalized Stokes equation over the symmetric tensors. The resulting system is the following. Seek  $(u_0, \dots, u_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ ,  $w \in Z_{1,m-1}$ , and  $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$  such that, for all  $j \in \{0, \dots, m-3\}$ ,

$$(7a) \quad (Du_j, Dv_j)_{L^2(\Omega)} = (u_{j+1}, Dv_j)_{L^2(\Omega)},$$

$$(7b) \quad (Du_{m-2}, Dv_{m-2})_{L^2(\Omega)} = (w, Dv_{m-2})_{L^2(\Omega)},$$

$$(7c) \quad (Dw, Dz)_{L^2(\Omega)} = (Dr_{m-2}, z)_{L^2(\Omega)},$$

$$(7d) \quad (Dr_{m-2-j}, Ds_{m-2-j})_{L^2(\Omega)} = (Dr_{m-3-j}, s_{m-2-j})_{L^2(\Omega)},$$

$$(7e) \quad (Dr_0, Ds_0)_{L^2(\Omega)} = (f, s_0)_{L^2(\Omega)}$$

for all  $(v_0, \dots, v_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ ,  $z \in Z_{1,m-1}$ , and  $(s_0, \dots, s_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ .

Hence, the following equivalence is valid.

**Proposition 4.** *If  $u \in H_0^m(\Omega)$  solves (6), then there exist  $(u_1, \dots, u_{m-2}) \in V_{1,1}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ ,  $w \in Z_{1,m-1}$ , and  $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$  such that, with  $u_0 := u$ , (7) is satisfied.*

*If, conversely,  $(u_0, \dots, u_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ ,  $w \in Z_{1,m-1}$ , and  $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$  solve (7), then  $u := u_0$  belongs to  $H_0^m(\Omega)$  and solves (6) with  $D^j u = u_j$  for all  $j \in \{1, \dots, m-2\}$  and  $D^{(m-1)}u = w$ .  $\square$*

**3.3. Discretization with standard finite elements.** The main advantage of the reformulation of (6) as the system (7) is that one can use standard finite elements for its discretization. In particular, for the Poisson-like problems in (7), standard  $H_0^1$ -conforming finite elements can be used without any restriction on the polynomial order. In the absence of lower-order terms, the equations in (7) even decouple into a finite sequence of second-order problems. The only equation in (7) that needs further investigation is (7c). This equation is similar to the classical Stokes system, but the side constraint is on the rotation (instead on the divergence for the classical Stokes equations). In three space dimensions, this results in a vector valued “pressure” variable. Moreover, the symmetry constraint implies further constraints for the space of multipliers in a saddle-point formulation. Hence, (7c) requires modifications of stable finite elements for the second-order generalized Stokes problem. As described in Sections 4–5 below, in the case of space dimensions 2 and 3 some of the known stable finite elements for the classical Stokes problem can be utilized.

#### 4. APPLICATION TO FOURTH-ORDER PROBLEMS

Throughout the remaining sections, the space dimension is  $d = 2$  or  $d = 3$ . If  $d = 3$ , it is assumed that  $\Omega$  is contractible.

This section is devoted to the application of the splitting to the biharmonic equation. It establishes a saddle-point formulation of the Stokes-type equation and studies discretization schemes.

**4.1. The mixed system.** The biharmonic equation

$$(8) \quad \Delta^2 u = f \quad \text{in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

is a classical model for the vertical deflection of a thin elastic plate subject to clamped boundary conditions. Its weak form is given by (6) for  $m = 2$ . In this case, the system from (7) seeks  $(u, w, r) \in H_0^1(\Omega) \times Z_{1,1} \times H_0^1(\Omega)$  such that

$$(9a) \quad (Du, Dv)_{L^2(\Omega)} = (w, Dv)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega),$$

$$(9b) \quad (Dw, Dz)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,1},$$

$$(9c) \quad (Dr, Ds)_{L^2(\Omega)} = (f, s)_{L^2(\Omega)} \quad \text{for all } s \in H_0^1(\Omega).$$



Similar systems were used in [11] for the analysis of the equations of the Reissner-Mindlin plate model in two space dimensions, which can be interpreted as a perturbed biharmonic equation.

For the numerical approximation of (9a) and (9c), any finite element method for Poisson's equation can be used. The next subsections focus on the approximation of (9b).

**4.2. The generalized Stokes equation.** This subsection is devoted to the saddle-point formulation of the problem

$$(10) \quad (Dw, Dz)_{L^2(\Omega)} = (g, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,1}$$

with given  $g \in L^2(\Omega; \mathbb{R}^d)$ . Recall that the space  $Z_{1,1}$  consists of the rotation-free vector fields in the first-order Sobolev space  $\mathbf{V} := V_{1,1}^{\mathbb{S}} = H_0^1(\Omega; \mathbb{R}^d)$ . For  $d = 3$ , the following space will be employed

$$(11) \quad H_0(\text{div}^0, \Omega) := \{q \in L^2(\Omega; \mathbb{R}^3) : \text{div } q = 0 \text{ in } \Omega \text{ and } q \cdot \nu|_{\partial\Omega} = 0\}.$$

Define the space  $\mathbf{Q}$  for the Lagrange multipliers as

$$\mathbf{Q} := \begin{cases} \{q \in L^2(\Omega; \mathbb{R}^2) : \int_{\Omega} q \, dx = 0\} & \text{if } d = 2, \\ H_0(\text{div}^0, \Omega) & \text{if } d = 3, \end{cases}$$

equipped with the  $L^2$  norm. The saddle-point problem reads as follows. Seek  $(w, p) \in \mathbf{V} \times \mathbf{Q}$  such that

$$(12) \quad \begin{aligned} (Dw, Dv)_{L^2(\Omega)} + (\text{rot } v, p)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)} & \text{for all } v \in \mathbf{V}, \\ (\text{rot } w, q)_{L^2(\Omega)} &= 0 & \text{for all } q \in \mathbf{Q}. \end{aligned}$$

System (12) can formally be written as

$$\begin{aligned} -\Delta w + \text{rot}^* p &= g \\ \text{rot } w &= 0. \end{aligned}$$

Since,  $\text{rot } \mathbf{V} \subseteq \mathbf{Q}$ , with the standard theory [7] it is readily verified that problems (10) and (12) are equivalent once the following inf-sup condition is proved.

**Proposition 5.** *Let  $d \in \{2, 3\}$  and assume that, if  $d = 3$ , the domain  $\Omega$  is contractible. There is a constant  $\beta > 0$  such that*

$$\beta \leq \inf_{q \in \mathbf{Q} \setminus \{0\}} \sup_{v \in \mathbf{V} \setminus \{0\}} \frac{(\text{rot } v, q)_{L^2(\Omega)}}{\|Dv\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}.$$

*Proof.* Provided the domain  $\Omega$  has vanishing first and second Betti number (which in 3D means that  $\Omega$  is contractible), the result follows from [22, Prop. A.1] or [1, Theorem 1.2]. For  $d = 2$ , the result is known as the Ladyzhenskaya lemma [2], and there are no restrictions on the topology of  $\Omega$ .  $\square$

**4.3. Finite elements for the generalized Stokes equation.** System (12) resembles the classical Stokes system. Indeed, in two space dimensions, (12) can be reformulated as Stokes system by means of a change of variables. Therefore, any stable finite element method can be used for the approximation of (12). In three dimensions, (12) differs from the Stokes equations, but the methods that are well-studied for the Stokes equations [7] can nevertheless be used for the discretization of the system. This is exemplified with two choices of discrete spaces: a continuous finite element method, namely a stabilized  $P_1$  element with discontinuous pressure, and the nonconforming  $P_1$  element.

4.3.1. *Notation.* In what follows,  $\mathcal{T}$  denotes a shape-regular simplicial triangulation of  $\Omega$ . The set of  $(d-1)$ -dimensional hyperfaces is denoted by  $\mathcal{F}$ . The space of piecewise polynomial functions of maximal degree  $k$  is denoted by  $P_k(\mathcal{T})$ . The space of functions with values in some finite dimensional space  $X$  whose components belong to  $P_k(\mathcal{T})$  is denoted by  $P_k(\mathcal{T}; X)$ . Furthermore, denote

$$S^k(\mathcal{T}) := P_k(\mathcal{T}) \cap H^1(\Omega) \quad \text{and} \quad S_0^k(\mathcal{T}) := P_k(\mathcal{T}) \cap H_0^1(\Omega).$$

Similarly,  $X$ -valued functions with components in  $S^k(\mathcal{T})$  (resp.  $S_0^k(\mathcal{T})$ ) are denoted by  $S^k(\mathcal{T}; X)$  (resp.  $S_0^k(\mathcal{T}; X)$ ).

4.3.2. *Continuous finite elements.* The finite element discretization of (12) is based on finite-dimensional subspaces  $\mathbf{V}_h \subseteq \mathbf{V}$  and  $\mathbf{Q}_h \subseteq \mathbf{Q}$  and seeks  $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$  such that

$$(13) \quad \begin{aligned} (Dw_h, Dv_h)_{L^2(\Omega)} + (\text{rot } v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} & \text{for all } v_h \in \mathbf{V}_h, \\ (\text{rot } w_h, q_h)_{L^2(\Omega)} &= 0 & \text{for all } q_h \in \mathbf{Q}_h. \end{aligned}$$

In two space dimensions, this system is equivalent to the Stokes equations after the change of coordinates  $(x_1, x_2) \mapsto (-x_2, x_1)$ . Therefore, any stable finite element pairing  $(\mathbf{V}_h, \mathbf{Q}_h)$  for the Stokes equations (for examples see, e.g., [7]) can be used for the numerical solution.

**Proposition 6.** *For  $d = 2$ , any pairing  $\mathbf{V}_h \subseteq \mathbf{V}$ ,  $\mathbf{Q}_h \subseteq \mathbf{Q}$  that is stable for the Stokes equations leads to a unique solution  $(w_h, p_h) \in (\mathbf{V}_h, \mathbf{Q}_h)$  to (13). It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - v_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - q_h\|_{L^2(\Omega)}.$$

□

In the case of three space dimensions, a conforming stable pairing requires the space  $\mathbf{Q}_h \subseteq H_0(\text{div}^0, \Omega)$  to be a subset of (pointwise) divergence-free vector fields. The (strong) incorporation of this constraint in a space of continuous discrete vector fields is difficult. The modification of existing Stokes elements with discontinuous pressure can lead to a stable pairing. This is exemplified with a stabilized  $P_1$ - $P_0$  element. The lowest-order Raviart-Thomas space of  $H(\text{div})$  conforming vector fields [7] in three dimensions is defined as

$$RT_0(\mathcal{T}) := \left\{ q \in H(\text{div}, \Omega) : \begin{array}{l} \text{for any } T \in \mathcal{T} \text{ there exist } a_T \in \mathbb{R}^3, b_T \in \mathbb{R} \\ \text{with } q|_T(x) = a_T(x) + b_T(x)x \quad \text{for all } x \in T \end{array} \right\}.$$

Let, for any face  $F \in \mathcal{F}$ ,  $b_F$  denote the cubic bubble function that vanishes on  $\partial F$  and satisfies  $\int_F b_F ds = 1$ . It satisfies, for any adjacent simplex  $T$ , the scaling

$$(14) \quad \|b_F\|_{L^2(T)} \approx \text{diam}(T)^{3/2}.$$

The space of three-dimensional vector-fields whose components are spanned by the functions  $(b_F : F \in \mathcal{F})$  is denoted by  $\mathcal{B}_3(\mathcal{F}; \mathbb{R}^3)$ . Define the following spaces of discrete functions

$$(15) \quad \mathbf{V}_h := S_0^1(\mathcal{T}; \mathbb{R}^3) \oplus \mathcal{B}_3(\mathcal{F}; \mathbb{R}^3) \quad \text{and} \quad \mathbf{Q}_h := RT_0(\mathcal{T}) \cap H_0(\text{div}^0, \Omega).$$

It is useful to note that  $\mathbf{Q}_h = P_0(\mathcal{T}; \mathbb{R}^3) \cap \mathbf{Q}$ , in particular elements of  $\mathbf{Q}_h$  are piecewise constant. This low-order pairing generalizes stabilized finite elements [7] for the Stokes equations.

**Proposition 7.** *Let  $d = 3$  and let  $\Omega$  be contractible. The system (13) with the choice (15) of discrete spaces has a unique solution  $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ . It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - v_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - q_h\|_{L^2(\Omega)}.$$

*Proof.* The proof follows the usual reasoning for stabilized Stokes elements. Let  $q_h \in \mathbf{Q}_h$ . According to Proposition 5, there exists  $v \in \mathbf{V}$  such that

$$\|q_h\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \lesssim (\text{rot } v, q_h)_{L^2(\Omega)}.$$

Let  $I_h : V \rightarrow V \cap P_1(\mathcal{T}; \mathbb{R}^3)$  denote a quasi-interpolation operator [8] satisfying, for any  $v \in V$  and any  $T \in \mathcal{T}$ , the stability estimate

$$(16) \quad \text{diam}(T)^{-1} \|v - I_h v\|_{L^2(T)} + \|DI_h v\|_{L^2(T)} \lesssim \|Dv\|_{L^2(\cup\{K \in \mathcal{T} : K \cap T \neq \emptyset\})}.$$

Define  $v_h \in \mathbf{V}_h$  by

$$v_h = I_h v + \sum_{F \in \mathcal{F}} b_F \int_F (v - I_h v) ds.$$

The combination of the scaling and stability properties (14) and (16) readily proves

$$\|Dv_h\|_{L^2(\Omega)} \lesssim \|Dv\|_{L^2(\Omega)}.$$

On the other hand, since the elements in  $\mathbf{Q}_h$  are piecewise constant and by the definition of  $v_h$ , one has for any  $T \in \mathcal{T}$  (with outward-pointing unit normal  $\nu_T$ ) that

$$(\text{rot}(v - v_h), q_h)_{L^2(T)} = \int_{\partial T} ((v - v_h) \wedge \nu_T) \cdot q_h ds = 0$$

(the wedge  $\wedge$  denotes the exterior product in  $\mathbb{R}^3$ ). Hence  $(\text{rot } v_h, q_h)_{L^2(\Omega)} = (\text{rot } v, q_h)_{L^2(\Omega)}$ . This establishes the discrete stability condition

$$\beta \lesssim \inf_{q_h \in \mathbf{Q}_h \setminus \{0\}} \sup_{v_h \in \mathbf{V}_h \setminus \{0\}} \frac{(\text{rot } v_h, q_h)_{L^2(\Omega)}}{\|Dv_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}}.$$

The stability and the error estimate follow from the usual theory for saddle-point problems [7].  $\square$

*Remark 8.* From the proof of Proposition 7 it can be seen that a smaller choice of  $\mathbf{V}_h$  with stabilization in the tangential directions only would also suffice for a stable pairing. This would correspond to an analogue of the SMALL element [7] for the usual Stokes system.

**4.3.3. Nonconforming  $P_1$  element.** Nonconforming methods where  $\mathbf{V}_h^{NC} \not\subseteq \mathbf{V}$  may have favourable stability properties with trial functions based on low-order polynomials. This subsection describes an application of the nonconforming  $P_1$  finite element [14] for the generalized Stokes system (12). Define the spaces

$$\mathbf{V}_h^{NC} := \left\{ v \in \mathcal{P}_1(\mathcal{T}; \mathbb{R}^d) \left| \begin{array}{l} v \text{ is continuous in the interior hyper-faces} \\ \text{midpoints and vanishes in the midpoints} \\ \text{of hyper-faces on the boundary} \end{array} \right. \right\}$$

and

$$\mathbf{Q}_h := \begin{cases} P_0(\mathcal{T}) \cap \mathbf{Q} & \text{if } d = 2, \\ RT_0(\mathcal{T}) \cap \mathbf{Q} & \text{if } d = 3. \end{cases}$$

Again, it will be made use of the fact that  $RT_0(\mathcal{T}) \cap \mathbf{Q} = P_0(\mathcal{T}; \mathbb{R}^3) \cap \mathbf{Q}$ . The  $\mathcal{T}$ -piecewise action of the differential operators  $D$  and  $\text{rot}$  is denoted by  $D_h$  and  $\text{rot}_h$ . The nonconforming finite element method seeks  $(w_h, p_h) \in \mathbf{V}_h^{NC} \times \mathbf{Q}_h$  such that

$$(17) \quad \begin{aligned} (D_h w_h, D_h v_h)_{L^2(\Omega)} + (\text{rot}_h v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} && \text{for all } v_h \in \mathbf{V}_h^{NC}, \\ (\text{rot}_h w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h. \end{aligned}$$

**Proposition 9.** *The system (17) has a unique solution  $(w_h, p_h) \in (\mathbf{V}_h^{NC}, \mathbf{Q}_h)$ . It satisfies the error estimate*

$$\begin{aligned} & \|D_h(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ & \lesssim \inf_{v_h \in \mathbf{V}_h^{NC}} \|D_h(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)} \\ & \quad + \sqrt{\sum_{T \in \mathcal{T}} \text{diam}(T)^2 \inf_{c \in \mathbb{R}} \|g - c\|_{L^2(T)}^2}. \end{aligned}$$

*Proof.* The nonconforming interpolation operator  $I_h^{NC} : \mathbf{V} \rightarrow \mathbf{V}_h^{NC}$  is defined for any  $v \in \mathbf{V}$  via the condition

$$\int_F (v - I_h^{NC}v) ds = 0 \quad \text{for all hyper-faces } F \in \mathcal{F}.$$

It is well-known [14] that, for any  $v \in \mathbf{V}$ , the function  $D_h I_h^{NC}v$  equals the  $L^2$  projection of  $Dv$  onto the piecewise constants. In particular, for any  $q_h \in \mathbf{Q}_h$ , any  $v \in \mathbf{V}$  satisfies  $(\text{rot } v, q_h)_{L^2(\Omega)} = (\text{rot}_h I_h^{NC}v, q_h)_{L^2(\Omega)}$ . This and  $\|D_h I_h^{NC}v\|_{L^2(\Omega)} \leq \|Dv\|_{L^2(\Omega)}$  establish the discrete inf-sup condition

$$\beta \lesssim \inf_{v_h \in \mathbf{V}_h^{NC} \setminus \{0\}} \sup_{q_h \in \mathbf{Q}_h \setminus \{0\}} \frac{(\text{rot}_h v_h, q_h)_{L^2(\Omega)}}{\|D_h v_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}}.$$

The a priori error estimate can be deduced with the techniques of [12] for  $d = 2$  and [16] for  $d = 3$ .  $\square$

**4.4. Numerical methods for the biharmonic equation.** This subsection describes the numerical approximation of the biharmonic system (9). Let  $\mathbf{U}_h \subseteq H_0^1(\Omega)$  be a finite-dimensional subspace and let  $\mathbf{V}_h, \mathbf{Q}_h$  be one of the admissible pairings from §4.3.2. Then the discrete problem seeks  $(u_h, w_h, p_h, r_h) \in \mathbf{U}_h \times \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{U}_h$

$$(18) \quad \begin{aligned} (Du_h, Dv_h)_{L^2(\Omega)} &= (w_h, Dv_h)_{L^2(\Omega)} && \text{for all } v_h \in \mathbf{U}_h, \\ (Dw_h, D\xi_h)_{L^2(\Omega)} + (\text{rot } \xi_h, p_h)_{L^2(\Omega)} &= (Dr_h, \xi_h)_{L^2(\Omega)} && \text{for all } \xi_h \in \mathbf{V}_h, \\ (\text{rot } w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h, \\ (Dr_h, Ds_h)_{L^2(\Omega)} &= (f, s_h)_{L^2(\Omega)} && \text{for all } s_h \in \mathbf{U}_h. \end{aligned}$$

Standard a priori error estimates for the Poisson problem together with the triangle inequality and the a priori error estimates from Propositions 6 and 7 give the following quasi-optimal error estimate

$$\begin{aligned} & \|D(u - u_h)\|_{L^2(\Omega)} + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|D(r - r_h)\|_{L^2(\Omega)} \\ & \lesssim \inf_{v_h \in \mathbf{U}_h, \xi_h \in \mathbf{V}_h, q_h \in \mathbf{Q}_h, s_h \in \mathbf{U}_h} [\|D(u - v_h)\|_{L^2(\Omega)} + \|D(w - \xi_h)\|_{L^2(\Omega)} \\ & \quad + \|p - q_h\|_{L^2(\Omega)} + \|D(r - s_h)\|_{L^2(\Omega)}]. \end{aligned}$$

It is emphasized that this error estimate can dispense with any regularity assumptions on the solution. In particular, if the solution is smooth enough such that  $Du, Dw, Dr, p$  have  $H^s$  regularity for some  $0 < s \leq 1$  (this is satisfied  $u \in H^{2+s}(\Omega)$  and the Poisson equation in  $\Omega$  has  $H^{1+s}(\Omega)$  regularity), this results in the following convergence rate

$$\begin{aligned} & \|D(u - u_h)\|_{L^2(\Omega)} + \|D(w - w_h)\|_{L^2(\Omega)} \\ & \quad + \|p - p_h\|_{L^2(\Omega)} + \|D(r - r_h)\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)} \end{aligned}$$

for the maximal mesh-size  $h$ . With standard duality techniques, the following convergence rates in weaker norms can be proven

$$\|u - u_h\|_{L^2(\Omega)} + \|w - w_h\|_{L^2(\Omega)} + \|r - r_h\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}.$$

Analogous estimates can be obtained for the nonconforming discretization.

*Remark 10.* With the nonconforming method from §4.3.3, the Hessian is approximated with piecewise derivatives of nonconforming  $P_1$  finite element functions. Hence, the method may be interpreted as a variant of the Morley finite element [13].

**4.5. More general fourth-order operators.** This section describes the application of the new methodology to more general fourth-order problems. For simplicity, constant material coefficients are assumed in the model problem: Let  $\delta \geq 0$  be a nonnegative number, let  $\gamma \in \mathbb{S}_d(2)$  be a symmetric and positive definite  $d \times d$  matrix and let  $\mathbf{B} \in \mathbb{S}_d(4)$  be a symmetric and positive definite fourth-order tensor. Given  $f \in L^2(\Omega)$ , the fourth-order model problem under consideration reads

$$\operatorname{div} \operatorname{div} \mathbf{B} D^2 u - \operatorname{div} \gamma D u + \delta u = f \text{ in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.$$

That is  $u \in H_0^2(\Omega)$  solves, for all  $v \in H_0^2(\Omega)$ ,

$$(\mathbf{B} D^2 u, D^2 v)_{L^2(\Omega)} + (\gamma D u, D v)_{L^2(\Omega)} + (\delta u, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

With the split from Section 3, this leads to the following equivalent system

$$(19) \quad \begin{array}{rcll} -\operatorname{div} \mathbf{B} D w + \gamma w & + \operatorname{rot}^* p & - D r & = 0 \\ \operatorname{rot} w & & & = 0 \\ & \delta u & - \Delta r & = f \\ \operatorname{div} w & - \Delta u & & = 0. \end{array}$$

The rows and columns have been rearranged for better readability. This is a symmetric problem in saddle-point form. The last row in (19) asks for  $w = D u$  while the third row implies  $-\operatorname{div}(D r) = f - \delta u$ . Hence, taking the divergence of the first row leads to  $\operatorname{div} \operatorname{div} \mathbf{B} D^2 u - \operatorname{div} \gamma D u + \delta u = f$ . Let

$$a(w, v) := (\mathbf{B} D w, D v)_{L^2(\Omega)} + (\gamma w, v)_{L^2(\Omega)} \quad \text{for any } w, v \in H_0^1(\Omega; \mathbb{R}^d).$$

The weak form of the the mixed system seeks  $(w, u, r) \in Z_{1,1} \times H_0^1(\Omega) \times H_0^1(\Omega)$  such that  $(z, s, v) \in Z_{1,1} \times H_0^1(\Omega) \times H_0^1(\Omega)$

$$\begin{array}{rcll} a(w, z) & - (D r, z)_{L^2(\Omega)} & & = 0 \\ & (\delta u, s)_{L^2(\Omega)} + (D r, D s)_{L^2(\Omega)} & & = (f, s)_{L^2(\Omega)} \\ - (w, D v)_{L^2(\Omega)} & + (D u, D v)_{L^2(\Omega)} & & = 0. \end{array}$$

The well-posedness of this system and the stability of the numerical methods suggested in 5.3 follow immediately, provided  $u$  and  $r$  are discretized with the same finite element spaces. The discretization from Subsection 4.4 is stable and leads to the a priori error estimates from that subsection.

*Remark 11.* For simplicity, the analysis of this paper focusses on the boundary condition  $u \in H_0^m(\Omega)$ , which in the context of plate theory corresponds to clamped boundary conditions. However, the arguments from the stability proofs apply equally well for more general boundary conditions. In particular, the discrete stability does not rely on additional smoothness assumptions. Subsection 6.3 shows an application for simply-supported plates.

## 5. APPLICATION TO SIXTH-ORDER PROBLEMS

This section deals with the case  $m = 3$ . The section is structured similarly as Section 4. The main difference lies in the identification of the multiplier space  $\mathbf{Q}$ , which requires proper side restrictions in order to allow for the inf-sup condition. For better readability, the following notation is used

$$\mathbb{M}_{d \times d} = \mathbb{R}^{d \times d} = \mathbb{M}_d(2) \quad \text{and} \quad \mathbb{S}_{d \times d} = \mathbb{S}_d(2).$$

Again, the convention (3) for the rotation is used. If  $v$  is a matrix-valued function, the rot operator is applied row-wise.

**5.1. The mixed system.** The triharmonic equation

$$(20) \quad -\Delta^3 u = f \quad \text{in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0 \quad \text{on } \partial\Omega$$

is given in weak form by (6) for  $m = 3$ . Differential operators of this type arise for example in mathematical models of silicon oxidation [4, 20]. The system from (7) seeks

$$(u_0, u_1, w, r_1, r_0) \in H_0^1(\Omega) \times H_0^1(\Omega; \mathbb{R}^d) \times Z_{1,2} \times H_0^1(\Omega; \mathbb{R}^d) \times H_0^1(\Omega)$$

such that

$$(21a) \quad (Du_0, Dv_0)_{L^2(\Omega)} = (u_1, Dv_0)_{L^2(\Omega)} \quad \text{for all } v_1 \in H_0^1(\Omega),$$

$$(21b) \quad (Du_1, Dv_1)_{L^2(\Omega)} = (w, Dv_1)_{L^2(\Omega)} \quad \text{for all } v_1 \in H_0^1(\Omega; \mathbb{R}^d),$$

$$(21c) \quad (Dw, Dz)_{L^2(\Omega)} = (Dr_1, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,2},$$

$$(21d) \quad (Dr_1, Ds_1)_{L^2(\Omega)} = (Dr_0, s_1)_{L^2(\Omega)} \quad \text{for all } s_1 \in H_0^1(\Omega; \mathbb{R}^d),$$

$$(21e) \quad (Dr_0, Ds_0)_{L^2(\Omega)} = (f, s_0)_{L^2(\Omega)} \quad \text{for all } s_0 \in H_0^1(\Omega).$$

The subsequent subsections will study the saddle-point formulation and numerical approximation of (21c).

**5.2. The generalized Stokes equation.** Let  $g \in L^2(\Omega; \mathbb{R}^d)$ . This subsection is devoted to the saddle-point formulation of the problem

$$(22) \quad (Dw, Dz)_{L^2(\Omega)} = (g, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,2}.$$

Recall that the space  $Z_{1,2}$  consists of the rotation-free symmetric tensor fields in the first-order Sobolev space  $\mathbf{V} := V_{1,2}^{\mathbb{S}} = H_0^1(\Omega; \mathbb{S}_{d \times d})$ . Thus, (22) is a generalized tensor-valued Stokes equation with a symmetry constraint. Recall the definition of the space  $H_0(\operatorname{div}^0, \Omega)$  from (11) let  $H_0(\operatorname{div}^0, \Omega; \mathbb{M}_{3 \times 3})$  denote the space of  $3 \times 3$  tensor fields whose rows belong to  $H_0(\operatorname{div}^0, \Omega)$ . Let  $\xi : \Omega \rightarrow \mathbb{R}$ ;  $x \mapsto x$  denote the identity mapping. Define the space of multipliers as

$$\mathbf{Q} := \begin{cases} \{q \in L^2(\Omega; \mathbb{R}^2) : \int_{\Omega} q \, dx = 0 \text{ and } \int_{\Omega} q \cdot \xi \, dx = 0\} & \text{if } d = 2, \\ \{q \in H_0(\operatorname{div}^0, \Omega; \mathbb{M}_{3 \times 3}) : \operatorname{tr} q = 0 \text{ a.e. in } \Omega\} & \text{if } d = 3. \end{cases}$$

For a stable saddle-point formulation, the following result is required.

**Proposition 12.** *Let  $d \in \{2, 3\}$  and assume that, if  $d = 3$ , the domain  $\Omega$  is contractible. For any  $q \in \mathbf{Q}$  there exists  $v \in \mathbf{V}$  with  $\operatorname{rot} v = q$  and  $\|Dv\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$ .*

*Proof.* Let  $d = 2$ . Let  $q \in \mathbf{Q}$ . By the classical Ladyzhenskaya lemma (applied row-wise) there exists a tensor field  $\tau \in H_0^1(\Omega; \mathbb{M}_{2 \times 2})$  with  $\operatorname{rot} \tau = q$  and  $\|D\tau\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$ . Its skew-symmetric part is determined by the entry  $\varphi := \tau_{12} - \tau_{21}$ . Recall the identity mapping  $\xi(x) = x$ . By the definition of  $\mathbf{Q}$  and integration by parts,

$$0 = (\operatorname{rot} \tau, \xi)_{L^2(\Omega)} = (\tau, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})_{L^2(\Omega)} = \int_{\Omega} \varphi \, dx.$$

Hence, by the Ladyzhenskaya lemma [2], there exists  $\alpha \in H_0^2(\Omega; \mathbb{R}^2)$  such that  $\varphi = \operatorname{rot} \alpha$  and  $\|\alpha\|_{H^2(\Omega)} \lesssim \|D\varphi\|_{L^2(\Omega)}$ . This implies

$$\operatorname{skw} D\alpha = \frac{1}{2} \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix} = \operatorname{skw} \tau.$$

With  $\text{rot } D\alpha = 0$ , this leads to

$$\text{rot sym } D\alpha = -\text{rot skw } D\alpha = -\text{rot skw } \tau.$$

Thus, from the split in symmetric and skew-symmetric parts one obtains

$$q = \text{rot}(\text{sym } \tau + \text{skw } \tau) = \text{rot}(\text{sym } \tau - \text{sym } D\alpha).$$

Hence, the assertion follows with  $v := \text{sym } \tau - \text{sym } D\alpha$ .

For the case  $d = 3$ , let  $q \in \mathbf{Q}$ . Then, by [22, Prop. A.1] or [1, Theorem 1.2] there exists  $\tau \in H_0^1(\Omega; \mathbb{M}_{3 \times 3})$  with  $\text{rot } \tau = q$  and  $\|D\tau\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$ . Abbreviate

$$\rho := \begin{pmatrix} \tau_{23} - \tau_{32} \\ \tau_{31} - \tau_{13} \\ \tau_{12} - \tau_{21} \end{pmatrix}.$$

Recall that  $q \in \mathbf{Q}$  and, thus,  $\text{tr } q = 0$  almost everywhere in  $\Omega$ . Thus, for any  $\phi \in H_0^1(\Omega)$ , the integration by parts plus elementary manipulations reveal, with the  $3 \times 3$  identity matrix  $I_{3 \times 3}$ , that

$$0 = (q, \phi I_{3 \times 3})_{L^2(\Omega)} = (\text{rot } \tau, \phi I_{3 \times 3})_{L^2(\Omega)} = (\rho, D\phi)_{L^2(\Omega)}.$$

This means that  $\text{div } \rho = 0$  in  $\Omega$  and, hence  $\rho \in H_0(\text{div}^0, \Omega) \cap H_0^1(\Omega; \mathbb{R}^3)$ . Thus, by [22, Prop. A.1] or [1, Theorem 1.2] there exists  $\alpha \in H_0^2(\Omega; \mathbb{R}^3)$  with  $\text{rot } \alpha = \rho$  and  $\|\alpha\|_{H^2(\Omega)} \lesssim \|D\rho\|_{L^2(\Omega)}$ . The representation (2) of  $\mathbf{rot}$  therefore shows

$$2 \text{skw } D\alpha = \mathbf{rot } \alpha = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{pmatrix} = -2 \text{skw } \tau.$$

Since  $\text{rot skw } D\alpha = -\text{rot sym } D\alpha$ , we infer

$$\text{rot } \tau = \text{rot sym } \tau + \text{rot skw } \tau = \text{rot sym } \tau + \text{rot sym } D\alpha.$$

The claim thus follows with the choice  $v := \text{sym } \tau + \text{sym } D\alpha$ .  $\square$

The saddle-point formulation seeks  $(u, p) \in \mathbf{V} \times \mathbf{Q}$  such that

$$(23) \quad \begin{aligned} (Du, Dv)_{L^2(\Omega)} + (\text{rot } v, p)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)} && \text{for all } v \in \mathbf{V}, \\ (\text{rot } u, q)_{L^2(\Omega)} &= 0 && \text{for all } q \in \mathbf{Q}. \end{aligned}$$

Proposition 12 and the inclusion  $\text{rot } \mathbf{V} \subseteq \mathbf{Q}$  imply that (23) is equivalent to (22), provided the domain  $\Omega$  is contractible for  $d = 3$ .

### 5.3. Finite elements for the generalized Stokes equation with symmetry.

This subsection presents finite elements for problem (23). Recall the notation for finite element spaces from §4.3.1.

5.3.1. *Continuous finite elements.* The finite element discretization of (23) is based on finite-dimensional subspaces  $\mathbf{V}_h \subseteq \mathbf{V}$  and  $\mathbf{Q}_h \subseteq \mathbf{Q}$  and seeks  $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$  such that

$$(24) \quad \begin{aligned} (Dw_h, Dv_h)_{L^2(\Omega)} + (\text{rot } v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} && \text{for all } v_h \in \mathbf{V}_h, \\ (\text{rot } w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h. \end{aligned}$$

In two space dimensions, some stable Stokes elements can be used ‘‘row-wise’’, but the additional constraints of  $\mathbf{V}$  and  $\mathbf{Q}$  must be taken into account. The following choice serves as an example how the established stabilized schemes can be adapted to this situation. Let, for any  $T \in \mathcal{T}$ ,  $b_T$  denote the cubic bubble function that vanishes on  $\partial T$  and satisfies  $\int_T b_T dx = 1$ . The space of symmetric tensors that are component-wise spanned by these functions is denoted by  $\mathcal{B}_3(\mathcal{T}; \mathbb{S}_{2 \times 2})$ . As a generalization of the stabilized  $P_1$  or MINI element [7] define the discrete spaces

$$\mathbf{V}_h := \mathbf{V} \cap (P_1(\mathcal{T}; \mathbb{S}_{2 \times 2}) \oplus \mathcal{B}_3(\mathcal{T}; \mathbb{S}_{2 \times 2})) \quad \text{and} \quad \mathbf{Q}_h := \mathbf{Q} \cap S^1(\mathcal{T}; \mathbb{R}^2).$$

“Row-wise” this is the MINI element, but with the symmetry constraint of  $\mathbf{V}$  and the integral constraint of  $\mathbf{Q}$ .

**Proposition 13.** *Let  $d = 2$ . Then the choice  $(\mathbf{V}_h, \mathbf{Q}_h)$  leads to a unique solution  $(w_h, p_h) \in (\mathbf{V}_h, \mathbf{Q}_h)$  to (24). It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)}.$$

*Proof.* The proof follows the usual design of a suitable Fortin interpolation (see, e.g., [7, §4.8.2]). The details are omitted.  $\square$

For three dimensions, the stabilized finite element suggested in (15) can be generalized to the tensor-case. Let  $RT_0(\mathcal{T}; \mathbb{M}_{3 \times 3})$  denote the space of  $3 \times 3$  tensor fields whose rows belong to  $RT_0(\mathcal{T})$ . Furthermore, let  $\mathcal{B}_3(\mathcal{F}; \mathbb{S}_{3 \times 3})$  denote the space of symmetric  $3 \times 3$  tensor fields whose rows belong to  $\mathcal{B}_3(\mathcal{F}; \mathbb{R}^3)$  (for the definition, cf. §4.3.2). Define the following spaces of discrete functions

$$(25) \quad \mathbf{V}_h := S_0^1(\mathcal{T}; \mathbb{S}_{3 \times 3}) \oplus \mathcal{B}_3(\mathcal{F}; \mathbb{S}_{3 \times 3}) \quad \text{and} \quad \mathbf{Q}_h := RT_0(\mathcal{T}; \mathbb{M}_{3 \times 3}) \cap \mathbf{Q}.$$

**Proposition 14.** *Let  $d = 3$  and let  $\Omega$  be contractible. The system (24) with the choice (25) of discrete spaces has a unique solution. It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)}.$$

*Proof.* The proof is analogous to that of Proposition 7.  $\square$

**5.3.2. Nonconforming  $P_1$  element.** Similar as in §4.3.3, the nonconforming  $P_1$  finite element can be used for the saddle-point problem (23), where the trial functions from §4.3.3 are taken row-wise and are combined with the additional symmetry and integral constraints on  $\mathbf{V}$  and  $\mathbf{Q}$ . The analysis is analogous to the foregoing paragraph and §4.3.3. The details are omitted.

**5.4. Numerical methods for triharmonic problems.** Analogous to Subsection 4.4, the mixed system (21) can be approximated with standard finite elements for the Poisson equations (21a), (21b), (21c), (21d). The Stokes-like system (21c) is approximated with one of the methods described in Section 5.3. Let  $\mathbf{U}_h \subseteq H_0^1(\Omega)$  be a finite-dimensional subspace and let  $\mathbf{V}_h, \mathbf{Q}_h$  be one of the admissible pairings from §5.3.1. Then the discrete problem seeks

$$(u_{h,0}, u_{h,1}, w_h, p_h, r_{h,1}, r_{h,0}) \in \mathbf{U}_h \times [\mathbf{U}_h]^d \times \mathbf{V}_h \times \mathbf{Q}_h \times [\mathbf{U}_h]^d \times \mathbf{U}_h$$

such that

$$\begin{aligned} (Du_{h,0}, Dv_{h,0})_{L^2(\Omega)} &= (u_{h,1}, Dv_{h,0})_{L^2(\Omega)} && \text{for all } v_{h,0} \in \mathbf{U}_h, \\ (Du_{h,1}, Dv_{h,1})_{L^2(\Omega)} &= (w_h, Dv_{h,1})_{L^2(\Omega)} && \text{for all } v_{h,1} \in [\mathbf{U}_h]^d, \\ (Dw_h, D\xi_h)_{L^2(\Omega)} + (\text{rot } \xi_h, p_h)_{L^2(\Omega)} &= (Dr_{h,1}, \xi_h)_{L^2(\Omega)} && \text{for all } \xi_h \in \mathbf{V}_h, \\ (\text{rot } w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h, \\ (Dr_{h,1}, Ds_{h,1})_{L^2(\Omega)} &= (Dr_{h,0}, s_{h,1})_{L^2(\Omega)} && \text{for all } s_{h,1} \in [\mathbf{U}_h]^d, \\ (Dr_{h,0}, Ds_{h,0})_{L^2(\Omega)} &= (f, s_{h,0})_{L^2(\Omega)} && \text{for all } s_{h,0} \in \mathbf{U}_h. \end{aligned}$$

Standard a priori error estimates for the Poisson problem together with the triangle inequality and the a priori error estimates from Propositions 13 and 14 give quasi-optimal error estimates for the quantity

$$\begin{aligned} &\sum_{j=0}^1 \left( \|D(u_j - u_{h,j})\|_{L^2(\Omega)} + \|D(r_j - r_{h,j})\|_{L^2(\Omega)} \right) \\ &\quad + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}. \end{aligned}$$



In particular, if the solution is smooth enough such that  $Du_0, Du_1, Dw, Dr_1, Dr_0, p$  have  $H^s$  regularity for some  $0 < s \leq 1$  (which is the case if  $u \in H^{3+s}(\Omega)$  and the Poisson equation in  $\Omega$  has  $H^{1+s}(\Omega)$  regularity), one obtains the convergence rate

$$\sum_{j=0}^1 \left( \|D(u_j - u_{h,j})\|_{L^2(\Omega)} + \|D(r_j - r_{h,j})\|_{L^2(\Omega)} \right) + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)}$$

for the maximal mesh-size  $h$ . With standard duality techniques, the following convergence rates in the  $L^2$  norms can be proven

$$\sum_{j=0}^1 \left( \|u_j - u_{h,j}\|_{L^2(\Omega)} + \|r_j - r_{h,j}\|_{L^2(\Omega)} \right) + \|w - w_h\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}.$$

Analogous estimates can be obtained for the nonconforming discretization. Also, more general systems can be considered. Let, for example,  $\delta \geq 0$  be a nonnegative number,  $\gamma \in \mathbb{S}_d(2)$  a symmetric and positive definite  $d \times d$  matrix,  $\mathbf{B} \in \mathbb{S}_d(4)$  be a symmetric and positive definite fourth-order tensor, and  $\mathcal{A} \in \mathbb{S}_d(8)$  a symmetric eighth-order tensor. Consider, for given  $f \in L^2(\Omega)$ , the equation

$$-\operatorname{div}^3 \mathcal{A} D^3 u + \operatorname{div}^2 \mathbf{B} D^2 u - \operatorname{div} \gamma D u + \delta u = f \text{ in } \Omega$$

subject to the boundary condition  $u = \partial u / \partial \nu = \partial^2 u / \partial \nu^2 = 0$  on  $\partial \Omega$ . The weak solution  $u \in H_0^3(\Omega)$  is characterized by

$$\begin{aligned} & (\mathcal{A} D^3 u, D^3 v)_{L^2(\Omega)} + (\mathbf{B} D^2 u, D^2 v)_{L^2(\Omega)} \\ & + (\gamma D u, D v)_{L^2(\Omega)} + (\delta u, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^3(\Omega). \end{aligned}$$

Following the lines of Section 3, this problem can be split into a symmetric mixed system and the finite element discretizations of Sections 5.3–5.4 yield quasi-optimal a priori error estimates.

## 6. NUMERICAL EXPERIMENTS

In this section, numerical computations with the new method are presented for fourth and sixth order problems. In all examples, uniform mesh-refinement is used. The mesh-size is quantified by the parameter  $h := \max_{T \in \mathcal{T}} \operatorname{diam}(T)$ . The rate  $\alpha$  of algebraic convergence  $h^\alpha$  for some error quantity  $\operatorname{error}(j)$  (with respect to the mesh-size  $h(j)$ ) in the  $j$ -th row of the table ( $j \geq 2$ ) is computed with the formula

$$\alpha = \frac{\log(\operatorname{error}(j-1)/\operatorname{error}(j))}{\log(h(j-1)/h(j))}.$$

The domains and their initial partitions are displayed in Figure 1 for 2D and Figure 2 for 3D.

**6.1. Biharmonic equation with smooth solution.** Consider the square domain  $\Omega = (0, 1)^2$  and the biharmonic equation (8) with the exact solution

$$u(x) = (x_1 - x_1^2)^2 (x_2 - x_2^2)^2 \quad \text{for } f = \Delta u.$$

For the  $H_0^1(\Omega)$  variables, standard  $P_1$  conforming finite elements are employed. The variables in the Stokes-like system are approximated with the MINI finite element [7]. Table 1 displays the convergence history. For this smooth solution, the optimal convergence order of  $h$  and  $h^2$  for the error  $\|Dw_h - D^2 u\|_{L^2(\Omega)}$  in the energy norm and the error  $\|u_h - u\|_{L^2(\Omega)}$  in the  $L^2$  norm, respectively, is observed.

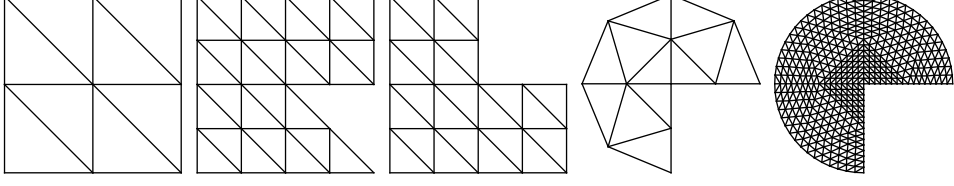


FIGURE 1. Initial meshes in 2D: the square, cusp, L, and sector domains. The last plot displays the triangulation of the sector domain after three refinements.

mesh-size $h$	$\ Dw_h - D^2u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
7.0711e-01	4.8604e-02	—	1.4042e-03	—
3.5355e-01	2.8576e-02	0.76	8.8575e-04	0.66
1.7678e-01	1.4373e-02	0.99	3.0992e-04	1.51
8.8388e-02	6.9823e-03	1.04	8.3667e-05	1.88
4.4194e-02	3.4409e-03	1.02	2.1225e-05	1.97
2.2097e-02	1.7098e-03	1.00	5.3134e-06	1.99
1.1049e-02	8.5257e-04	1.00	1.3272e-06	2.00
5.5243e-03	4.2576e-04	1.00	3.3152e-07	2.00

TABLE 1. Convergence history for the biharmonic equation on the square.

6.2. **Clamped plate.** Consider the equation

$$\Delta^2 u + u = f \quad \text{in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

in the the planar cusp domain  $\Omega = (-1, 1)^2 \setminus (\text{conv}\{(0, 0), (1, -1), (1, 0)\})$ . Define  $\omega := 7\pi/4$  and  $\alpha := 0.50500969$  as a noncharacteristic root of  $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$ . The exact singular solution from [18, p. 107] reads in polar coordinates

$$u(r, \theta) = (r^2 \cos^2 \theta - 1)^2 (r^2 \sin^2 \theta - 1)^2 r^{1+\alpha} g(\theta)$$

for

$$(26) \quad g(\theta) = \left[ \frac{\sin((\alpha - 1)\omega)}{\alpha - 1} - \frac{\sin((\alpha + 1)\omega)}{\alpha + 1} \right] (\cos((\alpha - 1)\theta) - \cos((\alpha + 1)\theta)) \\ - \left[ \frac{\sin((\alpha - 1)\theta)}{\alpha - 1} - \frac{\sin((\alpha + 1)\theta)}{\alpha + 1} \right] (\cos((\alpha - 1)\omega) - \cos((\alpha + 1)\omega)).$$

The right-hand side  $f$  is computed according to the exact solution. The derivation of the mixed system follows Subsection 4.5 with the identity tensor  $\mathbf{B}$  and  $\gamma = 0$ ,  $\delta = 1$ . As in the previous example, standard  $P_1$  conforming finite elements are used for the  $H_0^1(\Omega)$  variables and the MINI finite element is used for the Stokes-like system. The convergence history of the energy error  $\|Dw_h - D^2u\|_{L^2(\Omega)}$  and the  $L^2$  error  $\|u_h - u\|_{L^2(\Omega)}$  is displayed in Table 2. The solution belongs to the space  $H^{2+\alpha}(\Omega)$  [5, 23] and the expected convergence order for the energy error is  $h^{0.505}$ . This rate is approached by the values in Table 2. The error in the  $L^2$  norm is observed to decrease with the double rate.

6.3. **Vibrations of a simply-supported plate.** Consider the planar L-shaped domain  $\Omega = (-1, 1)^2 \setminus [0, 1]^2$ . The fundamental frequency of a vibrating thin elastic plate subject to simply-supported boundary conditions is described by the first eigenvalue  $\lambda$  of

$$\Delta^2 u = \lambda u \quad \text{in } \Omega \quad \text{and} \quad u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

mesh-size $h$	$\ Dw_h - D^2u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
7.0711e-01	3.0092e+00	—	1.6664e-01	—
3.5355e-01	1.8534e+00	0.69	8.1044e-02	1.04
1.7678e-01	1.0189e+00	0.86	2.6797e-02	1.59
8.8388e-02	5.7615e-01	0.82	8.2287e-03	1.70
4.4194e-02	3.4537e-01	0.73	2.7135e-03	1.60
2.2097e-02	2.1886e-01	0.65	1.0120e-03	1.42
1.1049e-02	1.4476e-01	0.59	4.2341e-04	1.25
5.5243e-03	9.8488e-02	0.55	1.9163e-04	1.14

TABLE 2. Convergence history for the clamped plate with cusp.

mesh-size $h$	$\lambda_h$	error $\frac{ \lambda - \lambda_h }{\lambda}$	rate
7.0711e-01	3.7632e+02	1.2984e+00	—
3.5355e-01	2.1643e+02	3.2185e-01	2.0123
1.7678e-01	1.8117e+02	1.0651e-01	1.5954
8.8388e-02	1.7093e+02	4.3946e-02	1.2772
4.4194e-02	1.6730e+02	2.1777e-02	1.0129
2.2097e-02	1.6573e+02	1.2180e-02	0.8383
1.1049e-02	1.6492e+02	7.2731e-03	0.7438
5.5243e-03	1.6447e+02	4.4861e-03	0.6971

TABLE 3. Convergence history for the first eigenvalue of the simply-supported L-shaped plate. The reference value is  $\lambda = 163.731$ .

understood in the weak sense in  $H_0^1(\Omega) \cap H^2(\Omega)$ . A reference value  $\lambda = 163.731$  for the first eigenvalue was computed with the adaptive Morley finite element method (see, e.g., [17]). This eigenvalue problem corresponds to the system from Subsection 4.5 where  $\mathbf{B}$  is the identity,  $\gamma = 0$  and  $\delta = 0$ , while  $f = \lambda u$ . The simply-supported boundary condition is realized by demanding homogeneous Dirichlet boundary conditions for the tangential component of  $w$  and for  $u$  on  $\partial\Omega$ . As in the previous example, standard  $P_1$  finite elements are combined with the MINI element. The convergence history of the first eigenvalue is displayed in Table 3. In this case, the solution belongs to the space  $H^{7/3}(\Omega)$  [5, 23] and the expected convergence order for the energy error is  $h^{1/3}$ . Accordingly, the convergence order of the eigenvalue error approaches  $h^{2/3}$ . This doubled rate of convergence is typical for symmetric eigenvalue problems, see, e.g., [27, 6]. It cannot be expected in general that the first eigenfunction is smoother than  $H^{7/3}(\Omega)$ . Indeed, this is a setting where some of the conventional mixed methods do not converge, see the examples in [10]. However, the method in this paper is stable also in the case of low smoothness.

**6.4. Tri-Laplacian on the square domain.** Let  $\Omega = (0, 1)^2$  be the unit square and consider the sixth-order problem (20) with exact solution [19]

$$u(x) = (x_1^2 + x_2^2)^{\frac{7}{4}} (x_1 - x_1^2)^3 (x_2 - x_2^2)^3.$$

The Poisson-type problems (21a), (21b), (21d), (21e) are approximated with standard conforming  $P_1$  finite elements and the Stokes-type system (21c) is discretized with the MINI element from §5.3.1. The convergence history of the energy error  $\|Dw_h - D^3u\|_{L^2(\Omega)}$  and the  $L^2$  error  $\|u_h - u\|_{L^2(\Omega)}$  is displayed in Table 4. The exact solution does not belong to  $H^5(\Omega)$ . However, the  $H^4$  regularity implies the convergence order  $h$  for this first-order scheme, as confirmed by the numerical experiment. The error in the  $L^2$  norm is observed to converge with order  $h^2$ .

mesh-size $h$	$\ Dw_h - D^3u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
7.0711e-01	3.8378e-02	—	8.9477e-05	—
3.5355e-01	3.3265e-02	0.20	6.9306e-05	0.36
1.7678e-01	1.9384e-02	0.77	3.6861e-05	0.91
8.8388e-02	9.3635e-03	1.04	1.2082e-05	1.60
4.4194e-02	4.4908e-03	1.06	3.2260e-06	1.90
2.2097e-02	2.2048e-03	1.02	8.1679e-07	1.98
1.1049e-02	1.0946e-03	1.01	2.0443e-07	1.99
5.5243e-03	5.4572e-04	1.00	5.1071e-08	2.00

TABLE 4. Convergence history for the Tri-Laplacian on the square domain.

mesh-size $h$	$\ Dw_h - D^3u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
7.6537e-01	2.4008e+00	—	9.4515e-03	—
4.2033e-01	1.6271e+00	0.64	6.5989e-03	0.59
2.2193e-01	9.2028e-01	0.89	2.8347e-03	1.32
1.1373e-01	5.2651e-01	0.83	9.3696e-04	1.65
5.7536e-02	3.1540e-01	0.75	2.9592e-04	1.69
2.8933e-02	1.9484e-01	0.70	9.7135e-05	1.62
1.4507e-02	1.2196e-01	0.67	3.4079e-05	1.51
7.2637e-03	7.6700e-02	0.67	1.2786e-05	1.41
3.6344e-03	4.8305e-02	0.66	5.0664e-06	1.33

TABLE 5. Convergence history for the Tri-Laplacian on the sector domain.

**6.5. Tri-Laplacian on the sector domain.** Let  $\Omega = \{(r, \theta) : r \in (0, 1), \theta \in (0, 3\pi/2)\}$  be the sector domain (defined in polar coordinates  $(r, \theta)$ ) and consider the sixth-order problem (20) with the exact solution given in polar coordinates as

$$u(r, \theta) = r^{8/3}(r^2 - r)^3 \sin(2\theta/3).$$

As in the previous example, standard conforming  $P_1$  finite elements and the MINI element from §5.3.1 are used. The curved boundary is approximated with polygonal domains. The convergence history is displayed in Table 5. The exact solution has regularity  $H^{11/3}(\Omega)$  and the expected convergence order is  $h^{2/3}$  for the energy error and  $h^{4/3}$  for the  $L^2$  error. The observed convergence rates correspond to these predictions.

**6.6. Bi-Laplacian on the cube domain.** Let  $\Omega = (0, 1)^3$  be the unit cube. Consider the biharmonic equation (8) with exact solution

$$u(x) = \prod_{j=1}^3 (x_j(1 - x_j))^2.$$

This problem is approximated with the stabilized  $P_1$  finite element from (15) for (9b), while (9a) and (9c) are discretized with standard  $P_1$  finite elements. Table 6 displays the convergence history of the energy error  $\|Dw_h - D^2u\|_{L^2(\Omega)}$  and the  $L^2$  error  $\|u_h - u\|_{L^2(\Omega)}$ . The convergence order for the energy error is  $h$ . The observed convergence order for the  $L^2$  error is between  $h$  and  $h^2$ .

**6.7. Bi-Laplacian on the tensor product L-domain.** Let

$$\Omega = \left( (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]) \right) \times (0, 1)$$

be the tensor product of a planar L-shaped domain with the unit interval. Consider the biharmonic equation (8). Define  $\omega := 3\pi/2$  and  $\alpha := 0.5444837$  as a noncharacteristic root of  $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$ . The exact singular solution is given in

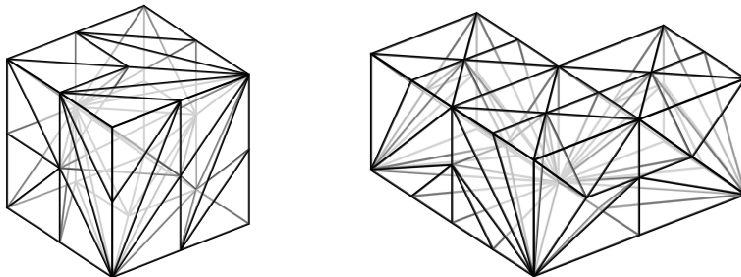


FIGURE 2. Initial meshes in 3D: cube and tensor product L-domain.

mesh-size $h$	$\ Dw_h - D^2u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
1.2247e+00	3.0507e-03	—	6.3507e-05	—
7.9057e-01	2.7227e-03	0.25	6.0334e-05	0.11
3.9528e-01	1.7855e-03	0.60	3.5170e-05	0.77
1.9764e-01	9.2940e-04	0.94	1.2177e-05	1.53
9.8821e-02	4.6011e-04	1.01	3.2859e-06	1.88

TABLE 6. Convergence history for the Bi-Laplacian on the cube domain.

mesh-size $h$	$\ Dw_h - D^2u\ _{L^2(\Omega)}$	rate	$\ u_h - u\ _{L^2(\Omega)}$	rate
1.4142e+00	7.2084e-01	—	2.1545e-02	—
7.9057e-01	6.5507e-01	0.16	1.9770e-02	0.14
3.9528e-01	4.2226e-01	0.63	1.0651e-02	0.89
1.9764e-01	2.2298e-01	0.92	3.4186e-03	1.63
9.8821e-02	1.1287e-01	0.98	9.1808e-04	1.89

TABLE 7. Convergence history for the Bi-Laplacian on the tensor product L-domain.

cylindrical coordinates by

$$u(r, \theta, z) = (z - z^2)^2 (r^2 \cos^2 \theta - 1)^2 (r^2 \sin^2 \theta - 1)^2 r^{1+\alpha} g(\theta)$$

for the function  $g$  from (26). Again, the stabilized  $P_1$  finite element is combined with standard  $P_1$  finite elements. The exact solution belongs to  $H^{2+\alpha}(\Omega)$  and the expected convergence order is  $h^{0.5444}$  for the energy error and  $h^{1.0888}$  for the error in the  $L^2$  norm. The empirical rates from Table 7 are slightly higher, which may be a pre-asymptotic phenomenon.

## 7. CONCLUSIVE REMARKS

- (a) The new splitting of polyharmonic equations presented in this work is stable, irrespective of the regularity of the solution. This stands in stark contrast with traditional mixed formulations which require additional regularity for a stable splitting. Therefore, the new method is also suitable for problems where the solution has only low regularity, see, e.g., the simply-supported plate problem from Subsection 6.3.
- (b) The new mixed formulation also covers the case of more general polyharmonic operators involving lower-order terms, see the outline in Subsection 4.5 and the numerical experiments.
- (c) The new formulation allows a finite element approximation of polyharmonic functions with lowest-order standard finite elements. The computer implementation is straightforward. The three-dimensional formulation requires

divergence-free functions for the pressure-like variable. This side constraint can be strongly incorporated with Raviart-Thomas finite elements.

- (d) The numerical experiments show that the pre-asymptotic regime is small. This suggests that the constants in the a priori error estimates are of moderate size.

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(D. Gallistl) INSTITUT FÜR NUMERISCHE SIMULATION, UNIVERSITÄT BONN, WEGELERSTRASSE 6,  
D-53115 BONN, GERMANY. PHONE: +49 228 73-2721 FAX: +49 228 73-3979  
*E-mail address:* gallistl(at)ins.uni-bonn.de